



8.2 Definition of linear maps and linear operators

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Introduction

- In this topic, we will
 - Define a linear map
 - Understand what it implies
 - Describe a test for linearity
 - Look at some examples
 - See that all finite-dimensional linear maps can be represented by matrices
 - Learn that the image of a linear map can be calculated by performing a matrix-vector multiplication





Background

- The two operations we can perform on vectors is:
 - Scalar multiplication: $\alpha \mathbf{u}$
 - **Vector** addition: $\mathbf{u} + \mathbf{v}$

Definition:

A map $A: \mathcal{U} \rightarrow \mathcal{V}$ is said to be a *linear map* if both

$$A(\alpha \mathbf{u}) = \alpha A\mathbf{u}$$

$$A(\mathbf{u}_1 + \mathbf{u}_2) = A\mathbf{u}_1 + A\mathbf{u}_2$$

- The response to a linear combination of inputs is equal to that same linear combination of the responses to the individual inputs

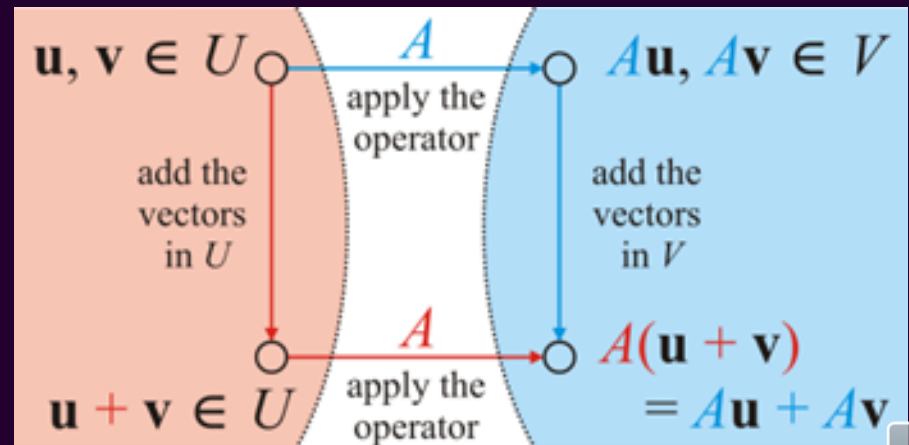
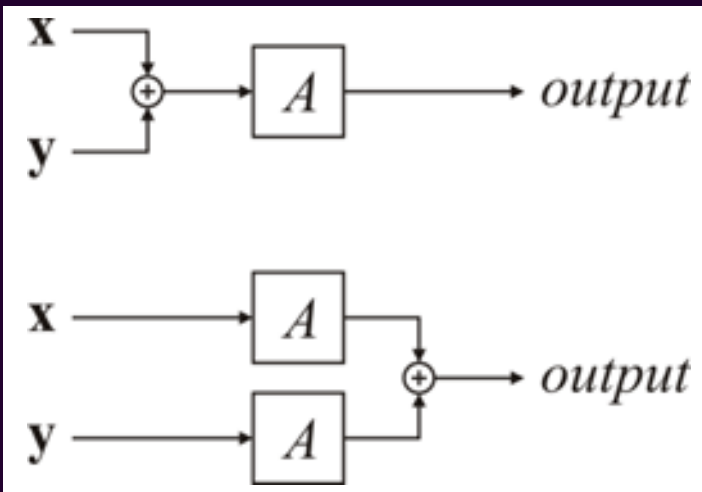
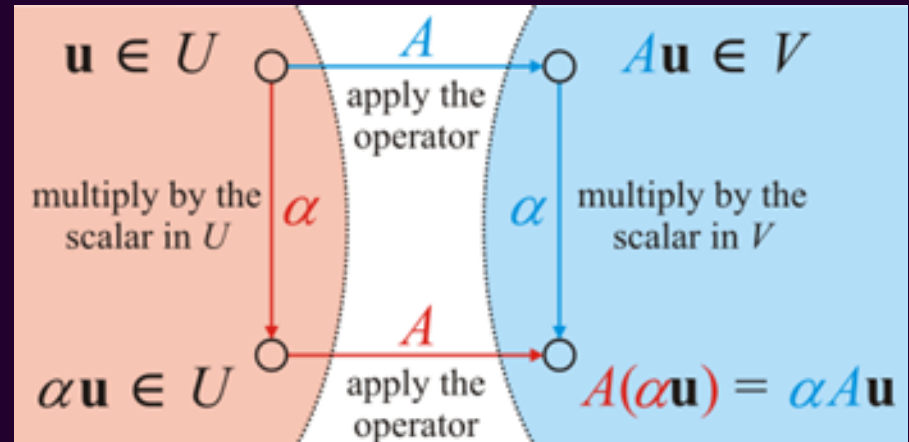
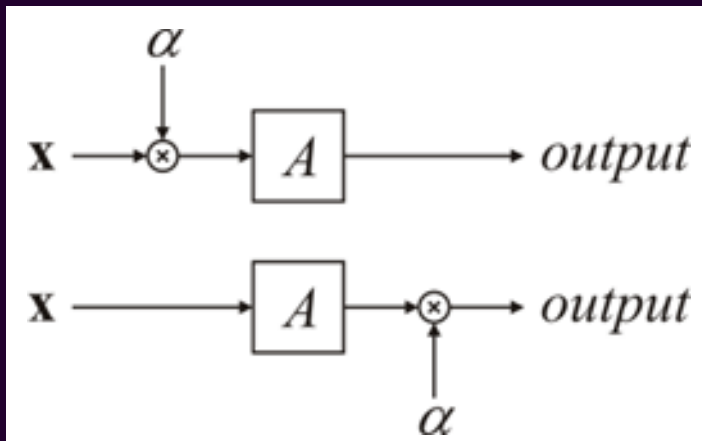
$$A(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = \alpha A\mathbf{u}_1 + \beta A\mathbf{u}_2$$





The superposition principle

- Visually, we can approach this in two ways:





Results

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is linear, then $A\mathbf{0}_{\mathcal{U}} = \mathbf{0}_{\mathcal{V}}$.

- That is, the zero vector in $\mathcal{U}$ is mapped to the zero vector in $\mathcal{V}$

Proof:

Let $\mathbf{u}$ be any vector in $\mathcal{U}$

Recall that $\mathbf{0}\mathbf{u} = \mathbf{0}_{\mathcal{U}}$

Thus, $A\mathbf{0}_{\mathcal{U}} = A(\mathbf{0}\mathbf{u}) = \mathbf{0} A\mathbf{u}$
 $= \mathbf{0}_{\mathcal{V}}$ ■

but $A\mathbf{u}$ is a vector in $\mathcal{V}$





Results

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is linear, then A maps lines onto lines or a single point.

Proof:

A line is of the form $\mathbf{u}_1 + \alpha \mathbf{u}_2$ in $\mathcal{U}$

Thus, $A(\mathbf{u}_1 + \alpha \mathbf{u}_2) = A\mathbf{u}_1 + \alpha A\mathbf{u}_2$

If $A\mathbf{u}_2 = \mathbf{0}_{\mathcal{V}}$, then $A(\mathbf{u}_1 + \alpha \mathbf{u}_2) = A\mathbf{u}_1$

Otherwise, $A\mathbf{u}_2 \neq \mathbf{0}_{\mathcal{V}}$, in which case, the result is a line in $\mathcal{V}$. ■





Results

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is linear and $A\mathbf{u}_1 = A\mathbf{u}_2 = \mathbf{v} \in \mathcal{V}$,
then $A(\alpha \mathbf{u}_1 + (1 - \alpha)\mathbf{u}_2) = \mathbf{v}$ for all $\alpha \in \mathbf{F}$.

Proof:

$$\begin{aligned} A(\alpha \mathbf{u}_1 + (1 - \alpha)\mathbf{u}_2) &= \alpha A\mathbf{u}_1 + (1 - \alpha)A\mathbf{u}_2 \\ &= \alpha \mathbf{v} + (1 - \alpha)\mathbf{v} \\ &= (\alpha + 1 - \alpha)\mathbf{v} = 1 \cdot \mathbf{v} = \mathbf{v} \quad \blacksquare \end{aligned}$$

- Note, this says: if there are two solutions, to $A\mathbf{u} = \mathbf{v}$,
then there are infinitely many solutions





Results

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is linear and $A\mathbf{u}_1 = A\mathbf{u}_2 = \mathbf{v} \in \mathcal{V}$,
then $A(\mathbf{u}_1 - \mathbf{u}_2) = \mathbf{0}_{\mathcal{V}}$.

Proof:

$$\begin{aligned} A(\mathbf{u}_1 - \mathbf{u}_2) &= A\mathbf{u}_1 - A\mathbf{u}_2 \\ &= \mathbf{v} - \mathbf{v} \\ &= \mathbf{0}_{\mathcal{V}} \blacksquare \end{aligned}$$





Background

Definition:

A map $A: \mathcal{U} \rightarrow \mathcal{U}$ is said to be a *linear operator* if the domain and codomain are the same vector space.

Definition

Given two linear maps, $A, B: \mathcal{U} \rightarrow \mathcal{V}$, we will say that $A = B$ if and only if $A\mathbf{u} = B\mathbf{u}$ for all $\mathbf{u} \in \mathcal{U}$.





Proving linearity

- You can show a map $A: \mathcal{U} \rightarrow \mathcal{V}$ is linear by either showing:
 1. $A(\alpha \mathbf{u}) = \alpha A\mathbf{u}$ for all $\mathbf{u} \in \mathcal{U}$ and all $\alpha \in \mathbf{F}$
 $A(\mathbf{u}_1 + \mathbf{u}_2) = A\mathbf{u}_1 + A\mathbf{u}_2$ for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$
 2. $A(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = \alpha A\mathbf{u}_1 + \beta A\mathbf{u}_2$ for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$ and all $\alpha, \beta \in \mathbf{F}$
 - You can show one or the other,
Requirement 1 is true if and only if Requirement 2 is true.
- To show that $A: \mathcal{U} \rightarrow \mathcal{V}$ is **not** linear, it is only necessary to show one of:
 1. $A(\mathbf{0}_{\mathcal{U}}) \neq \mathbf{0}_{\mathcal{V}}$
 2. $A(\alpha \mathbf{u}) \neq \alpha A(\mathbf{u})$ for one $\mathbf{u} \in \mathcal{U}$ and one $\alpha \in \mathbf{F}$
 3. $A(\mathbf{u}_1 + \mathbf{u}_2) \neq A\mathbf{u}_1 + A\mathbf{u}_2$ for two specific $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$





Examples

- Let $A: \mathbf{R}^2 \rightarrow \mathbf{R}^3$ be defined as $A\mathbf{u} = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 \begin{pmatrix} 0.5 \\ 1.2 \\ 0.9 \end{pmatrix} + u_2 \begin{pmatrix} 0.8 \\ 2.3 \\ 1.7 \end{pmatrix}$

- Now, $A(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = (\alpha u_{1,1} + \beta u_{2,1}) \begin{pmatrix} 0.5 \\ 1.2 \\ 0.9 \end{pmatrix} + (\alpha u_{1,2} + \beta u_{2,2}) \begin{pmatrix} 0.8 \\ 2.3 \\ 1.7 \end{pmatrix}$
$$= \alpha u_{1,1} \begin{pmatrix} 0.5 \\ 1.2 \\ 0.9 \end{pmatrix} + \beta u_{2,1} \begin{pmatrix} 0.5 \\ 1.2 \\ 0.9 \end{pmatrix} + \alpha u_{1,2} \begin{pmatrix} 0.8 \\ 2.3 \\ 1.7 \end{pmatrix} + \beta u_{2,2} \begin{pmatrix} 0.8 \\ 2.3 \\ 1.7 \end{pmatrix}$$
$$= \alpha \left(u_{1,1} \begin{pmatrix} 0.5 \\ 1.2 \\ 0.9 \end{pmatrix} + u_{1,2} \begin{pmatrix} 0.8 \\ 2.3 \\ 1.7 \end{pmatrix} \right) + \beta \left(u_{2,1} \begin{pmatrix} 0.5 \\ 1.2 \\ 0.9 \end{pmatrix} + u_{2,2} \begin{pmatrix} 0.8 \\ 2.3 \\ 1.7 \end{pmatrix} \right)$$
$$= \alpha A\mathbf{u}_1 + \beta A\mathbf{u}_2$$





Examples

- Let $B: \mathbf{R}^4 \rightarrow \mathbf{R}^2$ be defined as

$$B\mathbf{u} = B \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = u_1 \begin{pmatrix} 3.7 \\ 9.2 \end{pmatrix} + u_2 \begin{pmatrix} 4.8 \\ 6.0 \end{pmatrix} + u_3 \begin{pmatrix} 0.5 \\ 3.4 \end{pmatrix} + u_4 \begin{pmatrix} 0.7 \\ 2.6 \end{pmatrix} + \begin{pmatrix} 8.9 \\ 1.2 \end{pmatrix}$$

- Well, $B\mathbf{0}_4 = B \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 3.7 \\ 9.2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 4.8 \\ 6.0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0.5 \\ 3.4 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0.7 \\ 2.6 \end{pmatrix} + \begin{pmatrix} 8.9 \\ 1.2 \end{pmatrix} = \begin{pmatrix} 8.9 \\ 1.2 \end{pmatrix}$

and $\begin{pmatrix} 8.9 \\ 1.2 \end{pmatrix} \neq \mathbf{0}_2$, so this B is not linear





Examples

- Let $C: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ be defined as $C\mathbf{u} = C \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 - u_2 \\ u_2 - u_3 \end{pmatrix}$

- Now, $C\mathbf{u} = C \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + u_3 \begin{pmatrix} 0 \\ -1 \end{pmatrix},$

so using the same approach as for A ,
we can show that C , too, is linear





Examples

- Let $D:\mathbf{R}^4 \rightarrow \mathbf{R}^4$ be defined as $D\mathbf{u} = D \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 3u_1 + 7u_2 - 9u_4 \\ 2u_2 - 4u_3 - 3u_4 \\ 6u_1 + 5u_3 + 4u_4 \\ 9u_1 - 6u_2 - 7u_3 \end{pmatrix}$

- Now, $D\mathbf{u} = \begin{pmatrix} 3u_1 + 7u_2 - 9u_4 \\ 2u_2 - 4u_3 - 3u_4 \\ 6u_1 + 5u_3 + 4u_4 \\ 9u_1 - 6u_2 - 7u_3 \end{pmatrix} = u_1 \begin{pmatrix} 3 \\ 0 \\ 6 \\ 9 \end{pmatrix} + u_2 \begin{pmatrix} 7 \\ 2 \\ 0 \\ -6 \end{pmatrix} + u_3 \begin{pmatrix} 0 \\ -4 \\ 5 \\ -7 \end{pmatrix} + u_4 \begin{pmatrix} -9 \\ -3 \\ 4 \\ 0 \end{pmatrix}$

so using the same approach as for A and C ,
we can show that D , too, is linear





Examples

- Let $E: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be defined as $E\mathbf{u} = E\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos(u_1) - 1 \\ \sin(u_2) \end{pmatrix}$
- Now, $E\mathbf{0}_2 = E\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(0) - 1 \\ \sin(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}_2$, so we can't do this easily...
- However, $E\left(2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = E\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \cos(2) - 1 \\ \sin(2) \end{pmatrix}$ and $2 \cdot E\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cos(1) - 2 \\ 2\sin(1) \end{pmatrix}$
 - But $1.6829 \approx 2\sin(1) \neq \sin(2) \approx 0.90930$
 - Thus, E is not linear





Examples

- Let $F: \mathbf{R}^1 \rightarrow \mathbf{R}^1$ be defined as $F\mathbf{u} = F(u_1) = (2u_1 + 1)$
- Now, $F\mathbf{0}_1 = (1) \neq \mathbf{0}_1$,
so F is not linear





Examples

- Let $G:\mathbf{R}^2 \rightarrow \mathbf{R}^2$ be defined as $G\mathbf{u} = G\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1^3 \\ u_2^3 \end{pmatrix}$
 - Now, $G\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, yet $G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $G\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$,
but $2G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 8 \\ 8 \end{pmatrix} = G\left(2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$
 - Thus, G is not linear





A general observation...

- For finite-dimensional vector spaces,
it seems that a mapping $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$ is linear if and only if A is defined as the entries of $\mathbf{u}$ being the coefficients of a linear combination of n vectors in $\mathbf{F}^m$
- That is, there exist n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that

$$A\mathbf{u} = u_1 \mathbf{v}_1 + \dots + u_n \mathbf{v}_n$$





A general observation...

- Can we go the other way?

Given a linear mapping $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$, does it necessarily define a linear combination of n vectors in $\mathbf{F}^m$ with coefficients from the vector $\mathbf{u}$ from $\mathbf{F}^n$

- That is, if A is linear, do there exist n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that

$$A\mathbf{u} = u_1 \mathbf{v}_1 + \cdots + u_n \mathbf{v}_n$$





A general observation...

- Solution:

- In the domain $\mathbf{F}^n$,

- let $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \dots, \hat{\mathbf{e}}_n$ be the canonical basis, so $\hat{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, etc.

- Define $\mathbf{v}_k = A\hat{\mathbf{e}}_k$, so given any vector $\mathbf{u} \in \mathbf{F}^n$,

- we see
$$\begin{aligned} A\mathbf{u} &= A(u_1\hat{\mathbf{e}}_1 + u_2\hat{\mathbf{e}}_2 + \dots + u_n\hat{\mathbf{e}}_n) \\ &= u_1A\hat{\mathbf{e}}_1 + u_2A\hat{\mathbf{e}}_2 + \dots + u_nA\hat{\mathbf{e}}_n \\ &= u_1\mathbf{v}_1 + u_2\mathbf{v}_2 + \dots + u_n\mathbf{v}_n \end{aligned}$$





Matrix representation of a linear map

- Now, given a linear map $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$, we will represent A as the $m \times n$ matrix

$$A = (\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n)$$

- Each column $\mathbf{a}_k \in \mathbf{F}^m$ and we will describe $A\mathbf{u}$ as the matrix-vector multiplication

$$A\mathbf{u} = (\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n) \mathbf{u} = u_1 \mathbf{a}_1 + u_2 \mathbf{a}_2 + \cdots + u_n \mathbf{a}_n$$





Matrix representation of a linear map

- It is also possible to think of matrix-vector multiplication as follow:

$$A\mathbf{u} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & a_{m,4} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{pmatrix}$$

$$a_{1,1}u_1 + a_{1,2}u_2 + a_{1,3}u_3 + a_{1,4}u_4 + \cdots + a_{1,n}u_n = v_1$$

$$a_{2,1}u_1 + a_{2,2}u_2 + a_{2,3}u_3 + a_{2,4}u_4 + \cdots + a_{2,n}u_n = v_2$$

$$a_{3,1}u_1 + a_{3,2}u_2 + a_{3,3}u_3 + a_{3,4}u_4 + \cdots + a_{3,n}u_n = v_3$$

$$a_{m,1}u_1 + a_{m,2}u_2 + a_{m,3}u_3 + a_{m,4}u_4 + \cdots + a_{m,n}u_n = v_m$$





Examples

- Recall $A:\mathbf{R}^2 \rightarrow \mathbf{R}^3$ was $A\mathbf{u} = A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 \begin{pmatrix} 0.5 \\ 1.2 \\ 0.9 \end{pmatrix} + u_2 \begin{pmatrix} 0.8 \\ 2.3 \\ 1.7 \end{pmatrix}$
 - The matrix representation is $A = \begin{pmatrix} 0.5 & 0.8 \\ 1.2 & 2.3 \\ 0.9 & 1.7 \end{pmatrix}$
 - For example, if $\mathbf{u} = \begin{pmatrix} 0.3 \\ -0.1 \end{pmatrix}$ then

$$A\mathbf{u} = \begin{pmatrix} 0.5 & 0.8 \\ 1.2 & 2.3 \\ 0.9 & 1.7 \end{pmatrix} \begin{pmatrix} 0.3 \\ -0.1 \end{pmatrix} = \begin{pmatrix} 0.07 \\ 0.13 \\ 0.10 \end{pmatrix}$$





Examples

- Recall $C: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ was $C\mathbf{u} = C \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 - u_2 \\ u_2 - u_3 \end{pmatrix}$

- The matrix representation is $C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

- For example, if $\mathbf{u} = \begin{pmatrix} 0.5 \\ 0.2 \\ -0.7 \end{pmatrix}$ then

$$C\mathbf{u} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.2 \\ -0.7 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.9 \end{pmatrix}$$





Examples

- Recall $D: \mathbf{R}^4 \rightarrow \mathbf{R}^4$ was $D\mathbf{u} = D \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 3u_1 + 7u_2 - 9u_4 \\ 2u_2 - 4u_3 - 3u_4 \\ 6u_1 + 5u_3 + 4u_4 \\ 9u_1 - 6u_2 - 7u_3 \end{pmatrix}$
 - The matrix representation is $D = \begin{pmatrix} 3 & 7 & 0 & -9 \\ 0 & 2 & -4 & -3 \\ 6 & 0 & 5 & 4 \\ 9 & -6 & -7 & 0 \end{pmatrix}$

– If $\mathbf{u} = \begin{pmatrix} 0.2 \\ -0.3 \\ 0.1 \\ 0.4 \end{pmatrix}$ then $D\mathbf{u} = \begin{pmatrix} 3 & 7 & 0 & -9 \\ 0 & 2 & -4 & -3 \\ 6 & 0 & 5 & 4 \\ 9 & -6 & -7 & 0 \end{pmatrix} \begin{pmatrix} 0.2 \\ -0.3 \\ 0.1 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -5.1 \\ -2.2 \\ 3.3 \\ 2.9 \end{pmatrix}$





Summary

- Following this topic, you now
 - Understand the definition of linearity
 - Know the test for linearity
 - Know also how to show a mapping is not linear
 - Know that the zero vector is always mapped to the zero vector by a linear mapping
 - Have observed a number of examples
 - Know that a finite-dimensional linear map can always be represented by a matrix
 - Understand matrix-vector multiplication





References

- [1] https://en.wikipedia.org/wiki/Linear_map





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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